Likelihood ratio and asymptotic consistency

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General likelihood method

- Suppose that $X = (X_1, ..., X_n)$ has density or frequency function $p(x|\theta)$ and we wish to test $H : \theta \in \Theta_0$ vs $K : \theta \in \Theta_1$. The test statistic we want to consider is the likelihood ratio given by

$$L(x) = \frac{\sup\{p(x;\theta) : \theta \in \Theta_1\}}{\sup\{p(x;\theta) : \theta \in \Theta_0\}}.$$

- Tests that reject H for large values of $L(x)$ are called likelihood ratio tests.
- Through this one can derive likelihood ratio tests in several important testing problems.
- Although the calculations differ from case to case, the basic steps are always the same.

Basic steps

1. Calculate the MLE $\hat{\theta}$ of θ.
2. Calculate the MLE $\hat{\theta}_0$ of θ where θ varies only over Θ_0.
3. Form $\lambda(x) = p(x, \hat{\theta})/p(x, \hat{\theta}_0)$.
4. Find distribution of $\lambda(X)$ and specify the size α likelihood ratio test through the test statistic $\lambda(X)$ and its $(1-\alpha)$th quantile – $\xi_{1-\alpha}$.
5. Use the quantile to reject H if $\lambda(x) > \xi_{1-\alpha}$.

Homework *Explain why these steps are equivalent to the test proposed on the previous slide.*

Homework *Take a simple one-parameter family of distributions and carry out these steps. If analytical solutions are not possible, support your solution by computational methods.*

Asymptotic justification of the likelihood procedures

- Asymptotic theory of statistics provides justification of the likelihood ratio procedures.
- In this theory, one studies **likelihood ratio stochastic process**

$$\theta \mapsto \lambda_n(X, \theta)$$

  when $n \to \infty$.
- Details will follow in the future lectures.

Consistency – an example of an asymptotic property

- Consistency is often referred to as the 0'th order asymptotics.
- Suppose that we have a sample $X_1, ..., X_n$ from P_θ, where $\theta \in \Theta$.
- We want to estimate a real number (or a vector) $q(\theta)$ by $\hat{q}_n(X_1, \dots, X_n)$.
- Consistency states

$$\lim_{n\to\infty} \hat{q}_n(X_1, \dots, X_n) \stackrel{P}{=} q(\theta).$$

- This asymptotics allows to study

$$\log(\lambda_n(X)) = \log(p_n(X, \hat{q}_n(X)) - \log(p_n(X, q(\theta))),$$

  which under H_0 converges to a Gaussian process centered at zero.
- This can establish distributional properties of $\hat{q}_n(\theta)$, which leads to asymptotic normality of estimators.