Functional Data Analysis
Lund University

Project 1: Functional bases.

Fourier bases, polynomial bases, wavelets, splines

Perform all requested tasks. In principle, you should be able to do within the timeframe of two hours of lab. If for any reason you will not be able to conclude your work in the lab, it is expected that you deliver a written report showing that you have completed the work outside of the lab session. In such a case, on a separate paper give answers to the questions, attach a printout of the code you have used and all graphs that you deem important for this lab session. Your grade of the lab will be based on this material. Some useful R-code that can help in completing Project 1 can be found in the course webpage.

Part One – Directly working with functional bases

In this part, you will learn to define orthonormal bases in a direct manner in R. Then you will utilize certain properties of the basis to decompose a signal into Fourier series. You will investigate the properties of the approximation by numerical methods and compare them with theoretical results.

1. In the lecture we have seen how one can directly use R to define a Fourier basis on a grid of interval $[0, 1]$. We have seen there the explicit code for the sin/cos basis. Check numerically that the computed elements of the basis are orthonormal.

2. Perform similar tasks for the sin basis and then for the cos basis (see the lecture for their definitions).

3. In the lecture we have used a test function to illustrate its decomposition to the Fourier components. Define your own test function that you would like to analyze - be adventurous. Evaluate its squared norm.

4. Evaluate the coefficients of orthonormal decomposition of your function in the trygonometric basis of your choice. Write explicitely mathematical form of the first coefficient and give some interpretation to it.

5. Plot the squares of the remaining coefficients against the frequencies to which they correspond – this is the so-called (energy) spectrum of the test function. Can you think about interpretation that can be attached to this graph? Do you observe any peak frequencies in the plots? Can you relate such peaks with the data?

6. Evaluate the error of the approximation of the test function by the finite sum of the trygonometric functions obtained from the decomposition. First do it directly and then do by using the originally computed norm of the function and the values of the coefficients. Does the 'Pythagorean theorem' is confirmed by the compuation?

7. Make the graph of the error as a function of the number of element used in the approximation. Based on this graph try to predict how many terms you would need to obtain half of the current minimal error.

8. Based on your prediction perform the decomposition of your test function with the larger number of the basis elements and report the obtained error of the improved prediction.

Part Two – Gram-Schmidt orthonormalization

Not always the bases are given in the convenient orthonormalized form. However by knowing inner products of the elements of a non-normalized basis one can obtain the orthonormalization by the procedure implemented in the code presented in the lecture. Here we will practice orthonormalization of functions by the means of this code.

1. Consider the monomials on interval $[0, 1]$ and of the order up to 10. Use `gso` function to orthonormalize them and present the plots of normalized functions.
2. Check for which order the algorithm fails due to difficulties in distinguishing the monomials.
3. Compare the results of numerical orthonormalization with the analytical form of the Legendre polynomials given in the lecture.
4. Repeat the previous inquiries but for the interval $[0, 2]$. Can you now evaluate analytically the inner products of the monomials? At which value of the order algorithm fails in this case, comment.
5. Evaluate the error of estimation of your test function from the previous part and compare with the error seen in the Fourier approximation.

Part Three – Working with packages for wavelets and splines

1. Download and activate the packages `"wavelets"` and `"splines"`. Use R-help feature to get information about the functions `dwt()` and `bs()`.

2. Find out how one can plot the wavelet basis and plot Haar basis and other wavelet basis.

3. For the Haar basis evaluate the coefficients of the expansion with the values obtained by `dwt`.

4. Decompose your test function using wavelets and evaluate the squared norm of the error.

5. Use the function `bs()` to plot the cubic B-spline basis on interval [0,1] with the midpoint as the knot point (include also the endpoints).

6. Apply the Gram-Schmidt method to orthogonalize the B-splines.

Part Four – Working with the fda package

The fda package facilitate in a unified way treatment of such basis as Fourier, B-splines, and some polynomial bases. In this part of the lab, you should learn basic feature of

- Download the fda package.
- Run the code given for the Canadian Weather in the lecture.
- Read Sections 3.1 and 3.2 of the textbook to learn how to use the fda package to construct Fourier basis and Fourier decomposition of the functions. Understand the concept of the functional basis object with the class name `basisfd`.
- Compare the features given by `create.fourier.basis` to the direct approach given in the first problem of this lab.
- Do the same for the spline bases by reading first Section 3.3 and then compare using package `splines` with `create.bspline.basis`.