# NOTES RELATED TO THE PROGAMS ON LAPLACE DISTRIBUTION AND STOCHASTIC PROCESSES BUILT UPON THEM

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## 1. Parameterizations of univariate Laplace

This section is based on [2]. The generalized Laplace laws are best described by their characteristic functions

$$\phi(u) = e^{i\delta u}\left(1 - i\mu u + \frac{\sigma^2 u^2}{2}\right)^{-\tau}, \tag{1}$$

where $\tau > 0$, $\delta, \mu \in \mathbb{R}$ and $\sigma > 0$. The case of $\tau = 1$ is referred to as asymmetric Laplace and $\mu = 0$ corresponds to generalized symmetric Laplace distributions.

*Remark* 1. Frequently $\nu = 1/\tau$ is used as the shape parameter.

Another location parameter $\lambda$ by centering the gamma variable in the following representation

$$Y = \sigma\sqrt{\Gamma}Z + \mu(\Gamma - \tau) + \lambda\ , \tag{2}$$

where

$$\lambda = \delta + \tau\mu. \tag{3}$$

Another scale and skeweness parameters $\psi$ and $\eta$ are given by

$$\begin{aligned} \psi &= \sqrt{\tau(\sigma^2+\mu^2)},\ \psi > 0, \\ \eta &= \frac{\mu}{\sqrt{\sigma^2+\mu^2}},\ -1 \le \eta \le 1, \end{aligned} \tag{4}$$

and we note the inverting relations

$$\begin{aligned} \sigma &= \frac{\sqrt{\tau(1-\eta^2)}}{\psi}, \\ \mu &= \frac{\psi\eta}{\sqrt{\tau}}, \\ \delta &= \lambda - \psi\eta\sqrt{\tau} \end{aligned} \tag{5}$$

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1.1. **New tail parameter.** The parameter $\tau > 0$ can be replaced by $\zeta > 0$:

$$\zeta = \frac{1}{1+\tau},\ \zeta \in [0,1) \tag{6}$$

with the inversion formulas

$$\tau = \frac{1}{\zeta} - 1, \tag{7}$$

This leads to the following relation between the original parameterization $(\delta, \sigma, \mu, \tau) \in \mathbb{R} \times [0,\infty) \times \mathbb{R} \times (0,\infty)$ and the final one $(\lambda, \psi, \eta, \zeta) \in \mathbb{R} \times [0,\infty) \times [-1,1] \times [0,1)$:

$$\lambda = \delta + \mu\tau, \tag{8}$$

$$\psi = \sqrt{(\sigma^2+\mu^2)\tau}, \tag{9}$$

$$\eta = \frac{\mu}{\sqrt{\sigma^2+\mu^2}}, \tag{10}$$

$$\zeta = \frac{1}{1+\tau} \tag{11}$$

and the inverse relation

$$\delta = \lambda - \eta\psi\sqrt{\frac{1-\zeta}{\zeta}}, \tag{12}$$

$$\sigma = \psi\sqrt{\frac{\zeta(1-\eta^2)}{1-\zeta}}, \tag{13}$$

$$\mu = \eta\psi\sqrt{\frac{\zeta}{1-\zeta}}, \tag{14}$$

$$\tau = \frac{1}{\zeta} - 1. \tag{15}$$

The following lines of Octave/Matlab code perform transition from one parameterization to another:

```
lambda=delta+mu*tau;

psi=sqrt((sigma^2+mu^2)*tau);

eta=mu/sqrt(mu^2+sigma^2);

zeta=1/(1+tau);

delta=lambda-eta*psi*sqrt((1-zeta)/zeta);

sigma=psi*sqrt(zeta*(1-eta^2)/(1-zeta));

mu=eta*psi*sqrt(zeta/(1-zeta));

tau=1/zeta-1;
```

1.2. **Tail dependent skeweness parameter $\xi$.** When $\tau$ tends to infinity ($\zeta$ tends to zero) the generalized Laplace distributions and tend to normal distributions irrespectively of $\eta$. This could lead to inconsistencies in the estimation of $\eta$. Consider for example $\eta = 1$. Given the case of small $\zeta$, negative sample skewness will be observed nearly half of the time. Similar results would hold for samples from a normal distribution ($\zeta = 0$). However, since the value of $\eta > 0$ is prohibited in the case of negative skewness, it would force an estimated value of $\eta$ to be negative. As a result, the estimation of $\eta$ in the case $\eta = 1$ would show some inconsistencies. A parameter that allows for a smooth transition between the case $\eta = 1$ and $\eta = -1$, when $\zeta$ tends to zero, remedies the deficiencies. Thus, we propose a new asymmetry controlling parameter

$$\xi = \operatorname{sgn}(\eta)\eta^2\zeta, \ -\zeta \le \xi \le \zeta \tag{16}$$

with the inversion formulas

$$\xi = \frac{\operatorname{sgn}(\mu)}{(1+\sigma^2/\mu^2)\,(1+\tau)} \tag{17}$$

and

$$\delta = \lambda - \psi \operatorname{sgn}(\xi)\frac{\sqrt{(1-\zeta)|\xi|}}{\zeta}, \tag{18}$$

$$\sigma = \psi\sqrt{\frac{\zeta - |\xi|}{1-\zeta}}, \tag{19}$$

$$\mu = \psi \operatorname{sgn}(\xi)\sqrt{\frac{|\xi|}{1-\zeta}}, \tag{20}$$

$$\tau = \frac{1}{\zeta} - 1. \tag{21}$$

The following lines of Octave/Matlab code perform transition from one parameterization to another:

```
lambda=delta+mu*tau;
psi=sqrt((sigma^2+mu^2)*tau);
xi=sign(mu)/((1+sigma^2/mu^2)*(1+tau));
zeta=1/(1+tau);
delta=lambda-psi*sign(xi)*sqrt((1-zeta)*abs(xi))/zeta;
sigma=psi*sqrt((zeta-abs(xi))/(1-zeta));
mu=psi*sign(xi)*sqrt(abs(xi)/(1-zeta));
```

```
tau=1/zeta-1;
```

## 2. Moving averages

This section is based on [1].

**2.1. Exponential kernels.** If we take for $m$ the Lebesgue measure in $\mathbb{R}^d$ divided by $\nu > 0$, we obtain an extra numerical parameter (with the standard value $\nu = 1$) so we can write $\mathcal{LI}(\mu, \sigma, \nu; f)$. The case of $f(x) = \mathbf{1}_{[0,1]}(x)$ corresponds to the generalized Laplace distributions. We can also fully parameterize such distributions by taking a family of parameterized kernels. An interesting one consists of kernels $f(x) = K \exp(-\beta|x|^\alpha)$, where $|x|$ is the Euclidean norm in $\mathbb{R}^d$ and $K$ is a normalization constant. It leads to a fully parametrical model. The proportionality constant $K$ is taken such that $\mathbb{V}(X) = (\mu^2 + \sigma^2)/\nu$ for all members of the family. For the one dimensional case we have

$$\int_{\mathbb{R}} \exp(-\beta|x|^\alpha)\ dx = 2\beta^{-1/\alpha}\Gamma\left(\frac{\alpha+1}{\alpha}\right)$$

and $f(x) = K(\alpha, \beta) \cdot e^{-\beta|x|^\alpha}$ , where

$$K^2(\alpha, \beta) = \frac{2^{1/\alpha-1}\beta^{1/\alpha}}{\Gamma\left(\frac{\alpha+1}{\alpha}\right)}.$$

Thus by using an explicit form of the integral of $f^k$:

$$\int f^k = \left(\frac{2}{k}\right)^{1/\alpha} \left(\frac{(2\beta)^{1/\alpha}}{2\Gamma\left(\frac{\alpha+1}{\alpha}\right)}\right)^{k/2-1},$$

we obtain explicit formulas for the moments, skewness and kurtosis in terms of the gamma function. We also observe that for large $\alpha$, the kernel is converging to $1/\sqrt{2}$ on $[-1, 1]$, and thus the distribution of the integral becomes generalized Laplace with the parameters equal to $(0, \mu/\sqrt{2}, \sigma/\sqrt{2}, \nu/2)$. Whereas for small $\alpha$, the kernel will approximate a constant function on the increasing support. Thus by virtue of the central limit theorem the integral converges in distribution to a normal distribution.

For numerical purposed it is good to have a finite interval $[-r_k, r_k]$ over which $f^2(x)$ integrates to $1-\epsilon$ for $\epsilon > 0$. We note that

$$\int_{[-r_k,r_k]^c} K^2(\alpha,\beta)e^{-2\beta|x|^\alpha}\ dx = 2K^2(\alpha,\beta)\int_{r_k}^{\infty} e^{-2\beta x^\alpha}\ dx$$
$$= 2\frac{K^2(\alpha,\beta)}{(2\beta)^{1/\alpha}}\int_{(2\beta)^{1/\alpha}r_k}^{\infty} e^{-u^\alpha}\ du.$$

On the other hand

$$\int_L^\infty e^{-x^\alpha}\ dx = \int_{L^\alpha}^\infty e^{-u}u^{1/\alpha-1}\ du$$
$$\le e^{-L^\alpha/2}\int_{L^\alpha}^\infty e^{-u/2}u^{1/\alpha-1}\ du$$
$$\le e^{-L^\alpha/2}2^{1/\alpha}\Gamma(1/\alpha).$$

Combining these two together we get

$$\int_{[-r_k,r_k]^c} f^2(x) \le 2\frac{K^2(\alpha,\beta)}{\beta^{1/\alpha}}\Gamma(1/\alpha)e^{-\beta r_k^\alpha}$$
$$= 2^{1/\alpha}\alpha e^{-\beta r_k^\alpha}.$$

Thus the desired $r_k$ can be defined as

$$r_k = \left(\log\left((2^{1/\alpha}\alpha/\epsilon)^{1/\beta}\right)\right)^{1/\alpha}$$

## References

[1] S. Åberg and K. Podgórski. A class of non-gaussian second order random fields. *Extremes*, 2010. Online available before print publication, doi:10.1007/s10687-010-0119-1.

[2] K. Podgórski and J. Wegener. Estimation for stochastic models driven by laplace motion. *Comm. Statist. Theory Methods*, 2010. (in press).

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